An Analytical Method for Direct Calculation of E & H-Field Patterns of Conductor-Backed Coplanar Waveguides

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***Abstract*—A new direct method of computing the electromagnetic field patterns surrounding the conductor-backed coplanar waveguide (CPW) structure is proposed. Analytical closed-form expressions describing the quasi-TEM field pattern in both the air and the dielectric substrate for conductor-backed CPW's are presented. This approach is based on a new technique which employs a series of inverse conformal mappings to transform a known field pattern from a rectangular structure back into the CPW structure in order to obtain its unknown field pattern directly. A computer program based on this method has demonstrated the speed at which the fields can be plotted compared to existing methods which require repetitive application. Graphical results of these field patterns are presented as a function of the CPW's geometry and dielectric substrate thickness. These field maps which have been directly drawn with true curvilinear squares enable the determination of power flow density, since the same power flows through each square. This direct method of characterizing the power flow density throughout the CPW structure could become an important design tool for the modeling of coplanar monolithic microwave integrated circuits (CMMIC's).**

I. Introduction

To understand how CPW's operate, the electric and magnetic fields (E and H) must be evaluated around them. Such a knowledge of fields requires maps or pictures of them showing field lines and equipotential surfaces. These plots gives us information about the field intensities, potential differences, energy, and current densities throughout the CPW structure. Previous methods of field plotting have involved such approaches as solving Laplace's equation which requires lengthy iterative processing [1]. These approaches like the relaxation method provide approximate solutions and only produce satisfactory accuracy after many repetitive applications. Hence, wherever possible it would be preferable to produce a direct solution for plotting the E and H field surrounding a given structure. Such methods are currently under investigation, where a direct analytical solution for the E-field distribution at the conductor surfaces of CPW's was presented recently [2]. Direct analytic formulas using conformal mapping techniques have previously been employed for designing CPW directional couplers [3], for calculating the parameters of a general broadside-coupled coplanar waveguide for (M)MIC applications [4], and for the design of slot-coupled directional couplers between doubled-sided substrate microstrip lines [5].

In this paper, a direct method of E and H field plotting involving inverse conformal mappings is proposed. This analytical method has successfully demonstrated a direct technique for accurately plotting the complete transverse field patterns surrounding the conductor-backed CPW's structure. In this work, the infinite domain surrounding the CPW is initially mapped to the interior of a rectangle in the image domain by a sequence of two intermediate conformal mappings originally employed by Ghione and Naldi [6]–[8]. An important extension of these work is to employ the same two conformal mappings to inverse transform the rectangle, with its known field distributions, back to the original infinite domain in order to directly plot both the E and H field throughout the CPW's structure. This involves solving the incomplete elliptic integrals which describe the series of conformal transformations, whose arguments are complex, in order to analytical express the CPW's transverse coordinates in terms of the finite image domain's coordinate values.

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II. Method of Analysis

The coplanar waveguide configuration to be analyzed is shown in Fig. 1(a), where the ground planes are assumed to be infinitely wide and perfectly conducting with negligible metallic conductor thickness. The central conductor, of width $2a$, is placed between the two upper ground planes, of spacing $2b$, which are located on a substrate of thickness h, with relative permittivity ε_r. In so far as the electromagnetic field is concerned, it is sufficient to consider only the right-half plane of the guiding structure as shown in Fig. 1(b), where a perfect magnetic surface exists at $x = 0$. A sequence of two conformal transformations may be employed to evaluate the capacitance per unit length of this section [6], as shown in Fig. 1(c) and Fig. 1(d). The fields inside the dielectric substrate will be considered initially, where a separate transformation for the air filled upper-half plane will be analyzed later in order to obtain a complete description of the transverse fields surrounding the

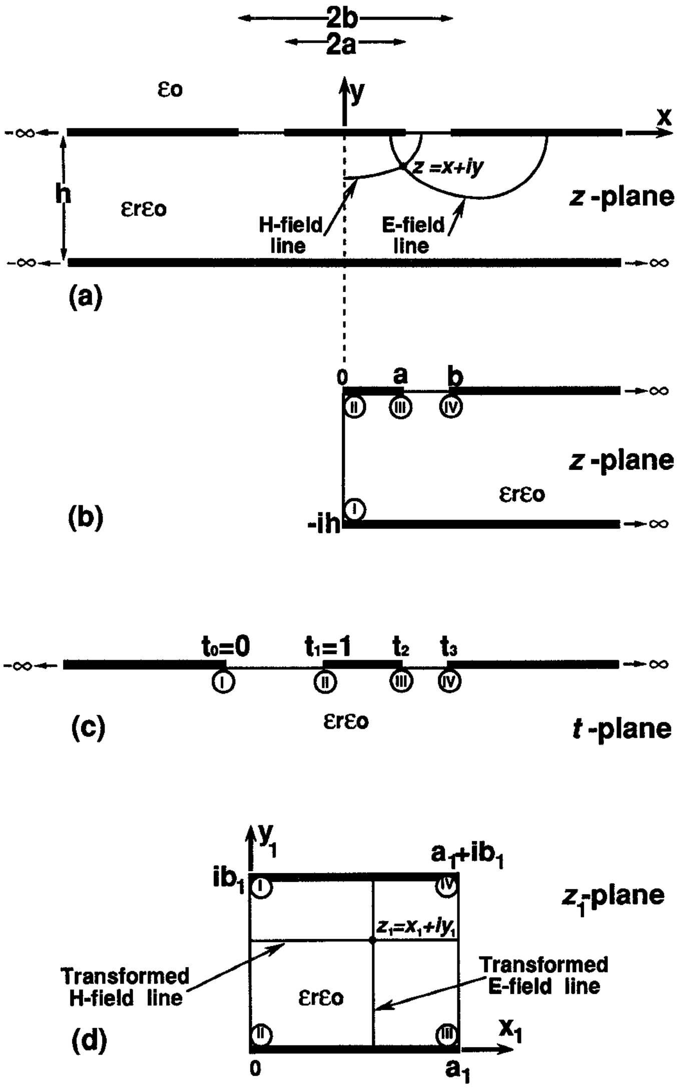


Fig. 1. Conformal transformations for evaluating the transverse electromagnetic field pattern for conductor-backed CPW's.

CPW geometry. An example of one E field and H field line is shown in Fig. 1(a) in the z plane, where its corresponding transformed E field and H field line is shown in Fig. 1(d) in the z_1 plane.

A. Field Construction Approach

In general the electrostatic field vector $\vec{E}$, generated from the potential between the transformed center conductor and ground planes in the finite image z_1 domain, can be expressed in terms of a potential function Φ, as

$$\vec{E} = -\nabla\Phi\,. \tag{1}$$

In this isotropic medium, the potential is a solution of the Laplace equation, written in the form using rectangular coordinates

$$\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial y_1^2} = 0\,. \tag{2}$$

However, since there is a uniform electric and magnetic field throughout the interior of the rectangle in the z_1 domain, the electromagnetic field components E_{y_1} and H_{x_1} are constant. Hence, the electrical potential has a simple solution, i.e., $\Phi(y_1) = y_1 E_{y_1}$. Thus, by inverse conformal mapping of these electromagnetic fields, from the z_1 domain back to the original z domain, we may directly construct the spatial field distributions surrounding the CPW geometry.

B. Conformal Mappings

By modeling the two slots between the center conductor and upper ground planes of the CPW as magnetic walls, various analytical expressions characterizing the CPW can be obtained [6]. This assumption is verified for practical line impedances where the relative slot width is not very large. The first conformal mapping transforms the transverse electromagnetic field of the right half of the dielectric substrate to the lower half of the t plane, using the Schwartz–Christoffel transformation:

$$t = \cosh^2\left(\frac{\pi z}{2h}\right) \qquad z = x + iy$$

where

$$t_2 = \cosh^2\left(\frac{\pi a}{2h}\right)$$

and

$$t_3 = \cosh^2\left(\frac{\pi b}{2h}\right). \tag{3}$$

For the second conformal mapping, the field in the lower half of the t plane is transformed to the finite image z_1 domain using:

$$z_1 = x_1 + iy_1 = \int_1^t \frac{A_1\,dt}{\sqrt{t(t-t_3)(t-t_2)(t-1)}} = F(\varphi_1, k_1) \tag{4}$$

where A_1 is a constant, and $z_1 = x_1 + iy_1$ defines the transformed point in the z_1 plane. Since the integral solution for the rectangle's coordinates in the z_1 plane, are $a_1 = A_1 K(k_1)$, and $ib_1 = A_1 i K(k_1')$, then the ratio a_1/b_1 of the rectangle in this z_1 plane can be evaluated from

$$\frac{a_1}{b_1} = \frac{K(k_1)}{K(k_1')} \tag{5}$$

where $K(k_1)$ is the complete elliptical integral of the first kind with modulus

$$k_1 = \frac{\tanh\left(\frac{\pi a}{2h}\right)}{\tanh\left(\frac{\pi b}{2h}\right)} \tag{6}$$

and where $k_1' = (1 - k_1^2)^{1/2}$ is the complementary modulus. The complex argument φ_1, of the incomplete elliptic integral $F(\varphi_1, k_1)$, is given by

$$\varphi_1 = \theta + i\psi\,. \tag{7}$$

The solution of this incomplete elliptic integral in terms of its real and imaginary parts, may be written as

$$F(\theta + i\psi, k_1) = F(\beta_1, k_1) + iF(\alpha_1, k_1') = x_1 + iy_1\,. \tag{8}$$

Equating the real and imaginary parts of $\sin\varphi_1$ in terms of the modified real amplitudes α and β, can be expressed by the following equations, respectively

$$\cosh\psi\sin\theta = \frac{\sin\beta_1\sqrt{1-k_1'^2\sin^2\alpha_1}}{\cos^2\alpha_1 + k_1^2\sin^2\beta_1\sin^2\alpha_1} \quad (9)$$

$$\cos\theta\sinh\psi = \frac{\sin\alpha_1\sqrt{1-k_1^2\sin^2\beta_1}}{\cos^2\alpha_1 + k_1^2\sin^2\beta_1\sin^2\alpha_1}. \quad (10)$$

From a solution of the Jacobian elliptic function (4), the complex argument φ_1 of the incomplete elliptic integral $F(\varphi_1, k_1)$, is defined in terms of the CPW's geometry as

$$\varphi_1 = \arcsin\sqrt{\frac{t_2(t-1)}{t(t_2-1)}} = \arcsin\left[\coth\left(\frac{\pi a}{2h}\right)\tanh\left(\frac{\pi z}{2h}\right)\right]. \quad (11)$$

For the purpose of evaluating z, (11) can be rearranged and expressed as

$$\begin{aligned}\tanh\left(\frac{\pi z}{2h}\right) &= \sin\varphi_1\tanh\left(\frac{\pi a}{2h}\right)\\ &= (\cosh\psi\sin\theta + i\cos\theta\sinh\psi)\tanh\left(\frac{\pi a}{2h}\right)\\ &= A + iB. \quad (12)\end{aligned}$$

Hence, the coordinates x, and y of the original z plane are expressed as

$$x = \frac{h}{\pi}\ln\left(\frac{(1+A)^2+B^2}{(1-A)^2+B^2}\right) \quad (13)$$

and

$$y = \frac{h}{\pi}\arctan\left(\frac{2B}{1-A^2-B^2}\right). \quad (14)$$

These expressions provides a description of the field pattern inside the CPW substrate as a function of the coordinates in the original z domain.

C. Field Pattern Above the Substrate

The method of field plotting above the substrate is similar to that of the above analysis, so only the results are summarized here. The transformation mapping the upper half of the z plane into the interior of another rectangle in the z_1 plane, is given by [9]

$$z_1 = x_1 + iy_1 = \int_0^z \frac{A_2\,dt}{\sqrt{(a^2-z^2)(b^2-z^2)}} = F(\varphi, k) \quad (15)$$

where $k = a/b$, and $\sin\varphi = z/a$. Hence, the coordinates x, and y of the original z plane are directly computed as

$$x = \frac{a\sin\beta\sqrt{1-k'^2\sin^2\alpha}}{\cos^2\alpha + k^2\sin^2\beta\sin^2\alpha} \quad (16)$$

and

$$y = \frac{a\sin\alpha\sqrt{1-k^2\sin^2\beta}}{\cos^2\alpha + k^2\sin^2\beta\sin^2\alpha}. \quad (17)$$

Thus, the component values of z in the upper-half plane are determined directly from the real and imaginary parts of $\sin\varphi_1$.

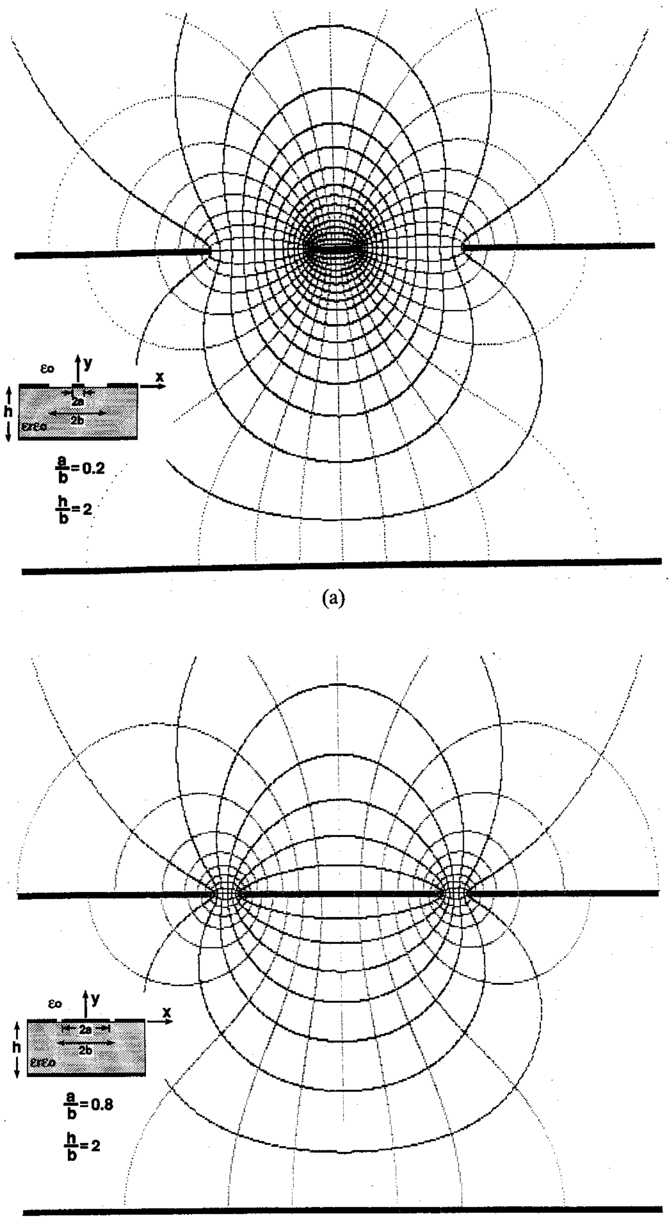
III. Graphical Results

Since the coordinates of z_1 in the finite image domain are expressed as $x_1 = F(\beta_1, k_1)$ and $y_1 = F(\alpha_1, k_1')$, an E field line can be constructed in the infinite z domain by keeping x_1 constant (thus, keeping β_1 constant) and varying y_1 from 0 to b_1 (thus varying α_1 from 0° to $\pi/2$). This allows the real and imaginary parts of $\sin\varphi_1$ to be computed which in turn enables the set of coordinates x, and y of the electric field line to be evaluated. Similarly, an H field can be constructed by keeping y_1 constant (thus, keeping α_1 constant) and varying x_1 from 0 to a_1 (thus, varying β_1 from 0° to $\pi/2$). In order to plot a field map which is made up of curvilinear squares where the same power is transmitted perpendicularly through each square, it is necessary to inverse transform a complete set of equally spaced E and H field lines back from the z_1 domain to the z domain. This requires each of the incomplete elliptic integrals, $F(\beta_1, k_1)$ and $F(\alpha_1, k_1')$ to be varied by equal steps, where each argument (β_1 and α_1) vary from 0° to $\pi/2$.

Shown in Fig. 2 are plots of field maps computed directly for various CPW dimensions. It can be clearly seen from Fig. 2(a), where $k = 0.2$, and $h/b = 2$, that the field is highly concentrated around the center conductor strip when the CPW transmission line has a relatively high characteristic impedance, Z_0. Fig. 2(b), shows the computed field map when $k = 0.8$, and $h/b = 2$ which demonstrates that most of the field is concentrated around the two conductor gaps, between the center conductor and upper ground planes, i.e., for CPW's with relatively low characteristic impedances. The variation in field distribution as a function of substrate thickness, h, is demonstrated in Fig. 3. As expected, there is less electric flux terminating on the lower ground plane of the CPW when the substrate thickness is increased, i.e., from $h/b = 1$, and $k = 0.5$ as shown in Fig. 3(a), to $h/b = 3$, and $k = 0.5$ as shown in Fig. 3(b). It is also noted that the field pattern in air appears to be independent of variations in the substrate thickness. This property only holds true so long as the two conductor spacings can be modelled as a perfect magnetic walls. This ensures that no electric field lines emanating into the air from the center conductor cross the air-dielectric boundary.

IV. Investigation Into Power Flow Density

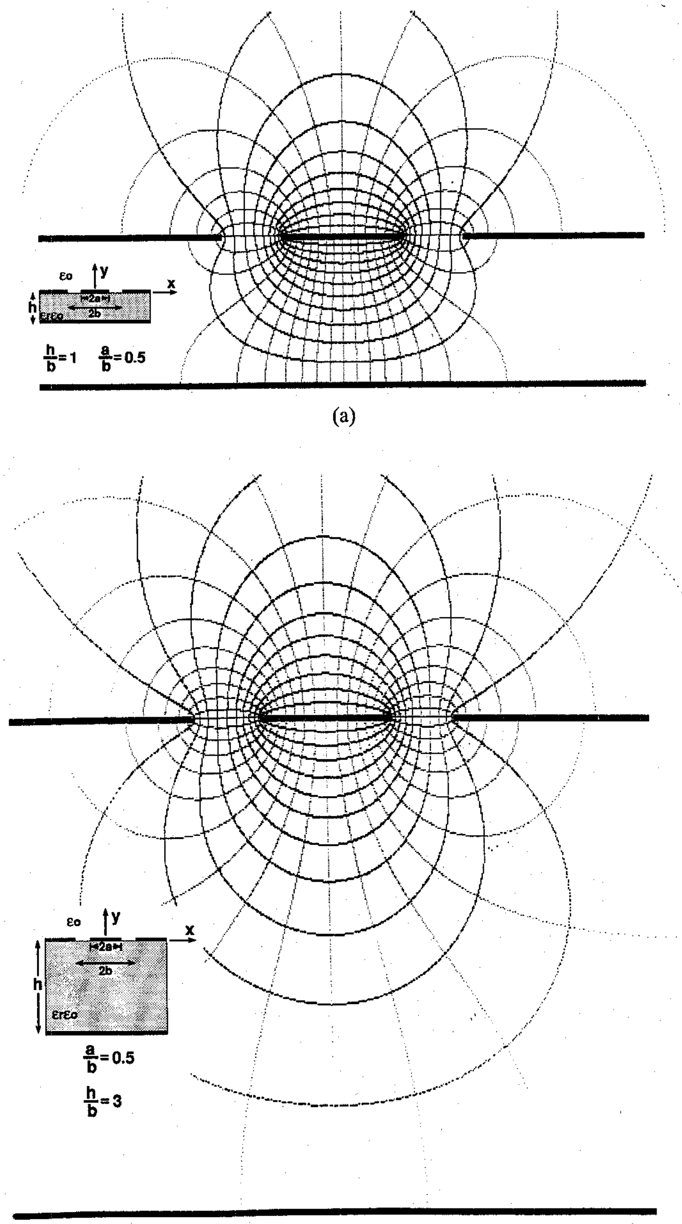
The vector product of the electric and magnetic field, known as the instantaneous Poynting vector, determines the perpendicular power flow density for the CPW transmission line. Since the characteristic impedance, Z_0, of the CPW is readily obtained by the earlier mentioned conformal mappings, the Poynting vector may be more conveniently expressed as equal to E^2/Z_0. Hence, the power flow density may be determined by knowing that it proportional to the square of the electric field E. The Poynting vector at any point z, can therefore be directly computed, which in turn would allow a power flow may to be drawn which consists of contours



(a)

(b)

Fig. 2. Spatial distribution of the transverse E and H-field pattern in both the dielectric substrate and in the air filled upper-half plane of the conductor-backed CPW for different values of center conductor spacings, (a) for $a/b = 0.2$ and $h/b = 2$, and (b) for $a/b = 0.8$ and $h/b = 2$.



(a)

(b)

Fig. 3. Spatial distribution of the transverse E and H-field pattern in both the dielectric substrate and in the air filled upper half plane of the conductor-backed CPW for different values of the substrate thickness, (a) for $a/b = 0.5$ and $h/b = 1$, and (b) for $a/b = 0.5$ and $h/b = 3$.

indicating regions of equal power flow density. Alternatively, the power flow density may be determined directly from the field map which is drawn with true curvilinear squares, where the same power flows through each square. Such a direct method of characterizing the power flow density throughput the CPW structure could become an important design tool for the realization of MMIC's which incorporate CPW transmission lines.

V. Conclusion

A direct method of constructing the transverse electromagnetic field lines surrounding the CPW's geometry has been proposed for the first time. The graphical results presented agree with previous iterative methods where the field plots vary with CPW's structural dimensions, as predicted. While we have limited ourselves to the case of grounded CPW's where the substrate is of finite thickness, the method shown here can be employed to analyze CPW's with infinitely thick substrates, with upper shielding, or without any lower ground plane.

We have described how a sequence of conformal mappings, in terms of complete and incomplete elliptic integrals, can be used to compute the spatial field distribution surrounding the CPW's geometry. These mappings overcome singularities at the conductor edges, since the normal field components in the image domain are smooth everywhere. The proposed new method does not require successive iterative processing in order to produce a field map and, consequently, has the advantage of being particularly appropriate for design analysis of CPW structures. A direct method of computing the power flow density by plotting the field pattern surrounding the CPW structure in terms of curvilinear squares is proposed. Thus, the potential for application of the above techniques is well suited to computer aided design analysis with relatively little programming effort.

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